

A k - ω MODEL FOR TURBULENTLY THERMAL CONVECTION IN STARS

YAN LI

National Astronomical Observatories/Yunnan Observatory, Chinese Academy of Sciences, P.O. Box 110, Kunming 650011, China
 Laboratory for Structure and Evolution of Celestial Objects, Chinese Academy of Sciences

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ABSTRACT

Both observations and numerical simulations show that stellar convective motions are composed of semi-regular flows of convective rolling cells and the fully developed turbulence. Although the convective rolling cells are crucial for the properties of the stellar convection that transports heat and mixes materials in the stellar interior, their contributions have not been included in turbulent convection models proposed up to now. We simplify the structure of the convective rolling cells as a cellular pattern moving circle by circle with different angular velocities around the center, estimating their typical size by solving approximately for the temperature difference over the stationary temperature background and their average shear of velocity by evaluating approximately their kinetic energy transformed by themselves working as thermal engines from the heat involved in the convective rolling cells. We obtain a steady state solution in the fully local equilibrium which is similar to what is obtained in the standard mixing-length theory, by applying such model assumptions to the standard k - ε model and properly choosing the model parameter $c_{\varepsilon 3}$. Accordingly, we propose a k - ω model to include the transport effect of turbulence in stars. Preliminary results of their applications to the sun and other stars with different masses and in different evolutionary stages show good agreements with results of the standard mixing-length theory and results of numerical simulations for the stellar convection.

Subject headings: stars: interiors — stars: evolution — convection — turbulence

1. INTRODUCTION

Thermal convection is a common phenomenon in the stellar interior. Convective flows are driven in stars by the thermal buoyancy, which results from unstable density stratifications. Both observations and numerical simulations show that convective motions are characterized by deterministic structures of different scales such as unsteady convective rolls and semi-regular convective cells created by large-scale instabilities of the buoyancy, along with the fully developed turbulence due to extremely high Reynolds numbers in the stellar convection zones. The bulk of a convection zone is stochastically filled up with numerous unsteady convective rolls pushing each other and leaving little space in between (Meakin & Arnett 2007; Kenjereš & Hanjalić 2009; Trampedach 2010). Close to the boundaries of the convective region, semi-regular convective cells jostle onto the surface and roughly line up seen for examples as the Rayleigh-Bénard convection (Kenjereš & Hanjalić 2000, 2006) and the solar granulation (Stein & Nordlund 1998; Freytag et al. 2012).

Such semi-deterministic and well-organized structures have already been recognized to have fundamental significance on the overall properties of the convection. From the dynamical point of view, the buoyancy drives the fluids continuously moving up and down alternatively in the convective rolling cells by doing the mechanical works respectively on the corresponding flows. Due to the equation of mass conservation, the flows form a cellular pattern with the warm fluids moving up in the center and the cold fluids moving down along the borders. Such a cellular motion creates shears of the velocities between the neighbouring layers in the convective rolling cells,

which contributes to the generation of turbulence. From the thermodynamic point of view, the cellular motion introduces inhomogeneous temperature perturbations over the stationary temperature background in the convective rolling cells, which results in extra fluxes transferring heat into or out of the boundaries of the convective cells. Therefore, each of the convective rolling cells runs as an individual heat engine, within which fluids absorbing heat at the higher temperature boundary while releasing heat at the lower temperature boundary, and the convective cell operates cycle by cycle to transform heat into the mechanical work, which is the energy supply of the cellular motions themselves in the convective rolling cells.

Thermal convection is an essential input physics for the stellar modelling. It not only carries heat outward from the stellar interior, but mixes different materials completely in the convective regions as well. In order to incorporate these effects of the convection in stellar models, various convection models have been proposed, including the most widely used mixing-length theory (Böhm-Vitense 1953, 1958) and more recently turbulent convection models (Xiong 1980, 1989; Deng et al. 2006; Canuto 1997; Canuto & Dubovikov 1998; Canuto et al. 2001; Li & Yang 2001). Turbulent convection models are based on the averaged hydrodynamic moment equations, and include many properties of turbulence such as the local effects of generation, dissipation, anisotropy, and the nonlocal effect of turbulent transport. However, all of them suffer from a common problem, i.e., the contributions from the convective rolling cells have not been considered up to now. Consequently, these models either do not result in a steady state solution in the fully local equilibrium, or have to adopt a length model of turbulence similar to what has been introduced in the standard

mixing-length theory.

In order to incorporate properly the effects of the convective rolling cells into the turbulent convection models, we have tried to find out the qualitative properties of the convective rolling cells in the stellar convection zone, by approximating the configuration of the convective rolling cells based on the results of the numerical simulations as a simple eddy structure with fluids moving circle by circle around the center of the eddy. The basic equations governing the motions of the convective rolling cells are given in Sect. 2. In Sect. 3, equations of the standard k - ε model for turbulence are introduced firstly, and then their steady state solutions in the fully local equilibrium are discussed. Based on the advanced turbulence models, we introduce new models for the rate of buoyancy production and for the turbulent heat flux with modifications suitable for the turbulent motions in the stellar convective regions. With the aid of above models, the temperature gradient in the convective region can be derived. In Sect. 4, we first estimate the typical size of the convective rolling cells by solving approximately for the temperature difference in a convective eddy, and then evaluate the shear production rate of turbulence by a model of the velocity shear. For vertical flows similar to those down-drafts seen in the numerical simulations of the stellar convection, we evaluate their development by introducing a decaying distance of the temperature difference along the vertical flow. In Sect. 5, we introduce a new choice of parameter $c_{\varepsilon 3}$ in the standard k - ε model, which leads to the well known properties of turbulence in the stellar convection zones. By comparing with the length model of turbulence in the standard mixing-length theory, we introduce a new macro-length model for the turbulent convection models. Based on above model assumptions, we obtain a steady state solution of the k - ε model in the fully local equilibrium, and compare it with the result of the mixing-length theory. In accordance with all model assumptions introduced above, we suggest a k - ω model for the stellar turbulent convection in Sect. 6, and apply it to the sun and some other stellar models with different masses in Sect. 7. We summarize in Sect. 8 our main conclusions, and discuss some important effects that do not yet properly incorporated in our stellar turbulent convection model.

2. EQUATIONS FOR MOTIONS OF CONVECTIVE ROLLING CELLS

We approximate the stellar convection zone as a plane parallel structure, and establish Cartesian coordinates (x, y, z) with the vertical direction z along the stellar radius.

We suppose that the stellar turbulent convection is composed of a more or less regular motion of convective rolling cells and a fully developed turbulent motion. The semi-regular motions of the convective rolling cells are determined by the Reynolds-averaged Navier-Stokes equation:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = g_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial U_i}{\partial x_j} - \overline{u_i u_j} \right). \quad (1)$$

In Eq. (1), the velocity of the convective flow U_i and the velocity fluctuation of turbulence u_i can be expressed

respectively as:

$$\begin{aligned} U_i &= (U, V, W), \\ u_i &= (u, v, w), \end{aligned} \quad (2)$$

g_i the component of gravitational acceleration in the x_i direction, p the pressure, ρ the density, and ν the molecular viscosity. Note that summation should be made over all three components if the subscript i appears twice in the same expression.

The stellar convection zone is as a whole in the static state, satisfying the equation of hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} = -\rho_0 g, \quad (3)$$

where ρ_0 is the stationary density distribution in the convection zone and the gravitational acceleration g is defined as:

$$g_i = (0, 0, -g). \quad (4)$$

The unstable stratification results in a density difference $\Delta\rho$ in the fluid, which is related to the temperature difference ΔT under the Boussinesq approximation:

$$\frac{\Delta\rho}{\rho_0} \approx -\beta\Delta T, \quad (5)$$

where T is the temperature and the thermodynamic coefficient β is defined by:

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p. \quad (6)$$

Substituting Eqs. (3) and (5) into Eq. (1), the convective motion in the steady state is described by:

$$U_j \frac{\partial U_i}{\partial x_j} = -\beta g_i \Delta T - \frac{\partial \overline{u_i u_j}}{\partial x_j}. \quad (7)$$

It can be noticed that the first term on the right hand side of Eq. (7) is the buoyancy, and the second term describes the resistance due to the Reynolds stress.

The Reynolds-averaged equation of energy conservation for the convective rolling cells can be expressed as:

$$\rho T \left(\frac{\partial s}{\partial t} + U_i \frac{\partial s}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left(\lambda \frac{\partial T}{\partial x_i} - F_C^i \right), \quad (8)$$

where s is the entropy of the stellar matter and F_C^i the convective heat flux in the x_i direction. The thermal conductivity of radiation λ is defined by:

$$\lambda = \frac{16\sigma T^3}{3\rho\kappa}, \quad (9)$$

where κ is the opacity of the stellar matter and σ the Stefan-Boltzmann constant.

Usually the total energy flux F is assumed to be a constant in the stellar convective envelope, which results in:

$$F = F_C - \lambda \frac{\partial T_0}{\partial z}, \quad (10)$$

where T_0 is the stationary temperature distribution in the convection zone and F_C the convective heat flux in the z direction. Substituting Eq. (10) into Eq. (8), it

TABLE 1
PARAMETERS' VALUES OF THE $k - \varepsilon$ MODEL

c_μ	σ_ε	$c_{\varepsilon 1}$	$c_{\varepsilon 2}$	References
0.09	1.3	1.44	1.92	Pope (2000)
		1.5	2.0	Tennekes (1989)

can be obtained that the temperature difference is approximately determined by:

$$\frac{\partial}{\partial x_i} \left(\lambda \frac{\partial \Delta T}{\partial x_i} \right) = \rho_0 c_p \frac{\partial \Delta T}{\partial x_i} U_i + \rho_0 T_0 \frac{\partial s}{\partial z} W, \quad (11)$$

where c_p is the specific heat at constant pressure. We omit hereafter the subscript "0" for the stationary density and temperature, which will give rise to no confusion on their meanings.

3. THE $k - \varepsilon$ MODEL FOR THE STELLAR TURBULENT CONVECTION

3.1. Equations of the $k - \varepsilon$ model with buoyancy modifications

The standard $k - \varepsilon$ model with buoyancy modifications consists of two equations (Pope 2000; Hossain & Rodi 1982):

$$\frac{Dk}{Dt} - \frac{\partial}{\partial x_i} \left(\nu_t \frac{\partial k}{\partial x_i} \right) = P + G - \varepsilon, \quad (12)$$

$$\frac{D\varepsilon}{Dt} - \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) = c_{\varepsilon 1} (P + c_{\varepsilon 3} G) \frac{\varepsilon}{k} - c_{\varepsilon 2} \frac{\varepsilon^2}{k}, \quad (13)$$

where $k = \frac{1}{2} \overline{u_i u_i}$ is the kinetic energy of turbulence, ε the dissipation rate of k , and D/Dt the co-moving derivative. On the left hand side of Eqs. (12) and (13), the standard gradient diffusion hypothesis is adopted to treat the turbulent transport process, and the turbulent viscosity ν_t is approximated according to the eddy viscosity model as:

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}. \quad (14)$$

On the right hand side of Eqs. (12) and (13), P represents the shear production rate of turbulent kinetic energy:

$$P = -\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}, \quad (15)$$

while G describes the contribution from the buoyancy:

$$G = -\beta g_i \overline{u_i \vartheta}, \quad (16)$$

where ϑ is the temperature fluctuation of turbulence. There are some model parameters in above equations, e.g. c_μ , σ_ε , $c_{\varepsilon 1}$, $c_{\varepsilon 2}$, and $c_{\varepsilon 3}$. Table 1 lists some choices of their values that are specified in applications of the standard $k - \varepsilon$ model. There are lots of controversy on the value of $c_{\varepsilon 3}$, and we shall discuss this in the following sections.

3.2. Steady state solutions in the fully local equilibrium

If the left hand sides of Eqs. (12) and (13) are equal to zero, the turbulence is in fully local equilibrium state. This results in a steady state solution:

$$\frac{P}{\varepsilon} = \frac{c_{\varepsilon 2} - c_{\varepsilon 1} c_{\varepsilon 3}}{c_{\varepsilon 1} - c_{\varepsilon 1} c_{\varepsilon 3}}, \quad (17)$$

$$\frac{G}{\varepsilon} = -\frac{c_{\varepsilon 2} - c_{\varepsilon 1}}{c_{\varepsilon 1} - c_{\varepsilon 1} c_{\varepsilon 3}}. \quad (18)$$

We use for simplicity the turbulence parameters suggested by Tennekes (1989) in Table 1, and define a new parameter as:

$$c_{\varepsilon 3} = 1 + \frac{c_{\varepsilon 2} - c_{\varepsilon 1}}{c_{\varepsilon 1}} c'_{\varepsilon 3} = 1 + \frac{1}{3} c'_{\varepsilon 3}. \quad (19)$$

As a result, it can be obtained that:

$$\frac{P}{\varepsilon} = 1 - \frac{1}{c'_{\varepsilon 3}}, \quad (20)$$

$$\frac{G}{\varepsilon} = \frac{1}{c'_{\varepsilon 3}}. \quad (21)$$

It is interesting to notice that only when $c'_{\varepsilon 3} > 0$ can the fully local equilibrium of turbulence appear in the convectively unstable region. Furthermore, the condition that $c'_{\varepsilon 3} > 1$ must be satisfied to ensure a positive rate of shear production.

3.3. Model for the rate of buoyancy production

As already pointed out by Kenjereš & Hanjalić (2000) that, in order to reproduce the ensemble roll pattern of convective cells, it is crucial to use an algebraic model for $\overline{u_i \vartheta}$ including all its production terms:

$$\left(1 + \frac{1}{y} \right) \overline{u_i \vartheta} = -c_\theta \frac{k}{\varepsilon} \left(\frac{T}{c_p} \frac{\partial s}{\partial x_j} \overline{u_i u_j} + \xi \frac{\partial U_i}{\partial x_j} \overline{u_j \vartheta} + \eta \beta g_i \overline{\vartheta^2} \right), \quad (22)$$

where the turbulent Péclet number y is defined as:

$$y = \frac{\rho c_p k^2}{\lambda \varepsilon}. \quad (23)$$

On the right hand side of Eq. (22), the first term in the brackets is the contribution from the stratification, the second one from the velocity shear of convective rolling cells, and the last one from the buoyancy. We introduce a modification of time scale from radiative dissipation on the left hand side (Canuto & Dubovikov 1998; Li & Yang 2001).

For the auto-correlation of temperature fluctuation $\overline{\vartheta^2}$ in Eq. (22), we adopt a local equilibrium approximation that assumes a complete balance between its production and dissipation:

$$\overline{\vartheta^2} = -c'_\theta \frac{k}{\varepsilon} \frac{T}{c_p} \frac{\partial s}{\partial x_i} \overline{u_i \vartheta}. \quad (24)$$

Solving for the velocity-temperature correlation $\overline{w \vartheta}$ in the z direction and neglecting some angle-dependent ingredients, we finally approximate the rate of buoyancy production by the following model:

$$G = -\frac{c'_\mu}{1 + y^{-1} + c'_\mu c_T \tau^2 N^2} \frac{k^2}{\varepsilon} N^2, \quad (25)$$

where the buoyancy frequency N is defined as:

$$N^2 = \beta g \frac{T}{c_p} \frac{\partial s}{\partial z}, \quad (26)$$

and the typical time scale of turbulence τ is defined as:

$$\tau = \frac{k}{\varepsilon}. \quad (27)$$

It can be noticed from Eq. (25) that the property of the stratification determines the effect of buoyancy contribution: $G > 0$ in a convection zone where $N^2 < 0$, while $G < 0$ in a stably stratified region where $N^2 > 0$.

3.4. Model of the convective heat flux

Based on similar arguments as discussed in deriving Eq. (25), we approximate the convective heat flux F_C by:

$$F_C = -\frac{\rho c_p}{\beta g} \frac{c'_\mu \tau_h k}{1 + y^{-1} + c'_\mu c_T \tau^2 N^2} N^2 \quad (28)$$

with a characteristic time scale τ_h that is often approximated by a model proposed by Nagano & Kim (1988) as:

$$\tau_h = \sqrt{\tau_\theta \tau}. \quad (29)$$

In Eq. (29), the kinetic time scale of turbulence τ is defined by Eq. (27), and the thermal time scale of turbulence τ_θ is approximated by:

$$\tau_\theta = \frac{\rho c_p}{\lambda} \frac{k^2}{\varepsilon}. \quad (30)$$

As a result, the convective heat flux is then expressed as:

$$F_C = -\frac{\lambda}{\beta g} \frac{c'_\mu y^{3/2}}{1 + y^{-1} + c'_\mu c_T \tau^2 N^2} N^2. \quad (31)$$

3.5. The temperature gradient in the stellar convection zone

In the stellar convective envelope, the temperature gradient ∇ can be obtained by substituting Eq. (31) into Eq. (10) as:

$$\nabla = \frac{d \ln T}{d \ln p} = \nabla_r + \frac{H_p}{\beta g T} \frac{c'_\mu y^{3/2}}{1 + y^{-1} + c'_\mu c_T \tau^2 N^2} N^2, \quad (32)$$

where ∇_r is the radiative temperature gradient assuming that the heat flux is solely transferred by radiation, and the local pressure scale height is defined as:

$$H_p = -\frac{dr}{d \ln p} = \frac{p}{\rho g}. \quad (33)$$

The buoyancy frequency N can be expressed as:

$$N^2 = -\frac{\beta g T}{H_p} (\nabla - \nabla_{ad}), \quad (34)$$

where the adiabatic temperature gradient ∇_{ad} is defined by:

$$\nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln p} \right)_s. \quad (35)$$

With the definition of the efficiency of convective heat transfer f as:

$$f = \frac{\nabla - \nabla_{ad}}{\nabla_r - \nabla_{ad}}, \quad (36)$$

the buoyancy frequency can be written as:

$$N^2 = -\frac{\beta g T}{H_p} (\nabla_r - \nabla_{ad}) f = -E f, \quad (37)$$

where E is a function of the stellar structure.

Directly solving for f from Eq. (32) will result in a singular solution. Instead, we introduce a new variable:

$$h = \frac{1 - f}{f}, \quad (38)$$

and Eq. (32) becomes:

$$(1 + y) h^2 + (1 + y - c'_\mu y^{5/2} - c'_\mu c_T E \tau^2 y) h - c'_\mu y^{5/2} = 0$$

with $A_1 = c'_\mu c_T E \tau^2 y + c'_\mu y^{5/2} - y - 1$. (39)

Equation (39) is a quadratic equation and its solution is:

$$h = \frac{A_1 + \sqrt{A_1^2 + 4(1 + y) c'_\mu y^{5/2}}}{2(1 + y)}. \quad (40)$$

It is easy to verify that h is always positive not only in the convection zone ($E > 0$) but also in the overshooting regions ($E < 0$).

4. SHEAR PRODUCTION RATE FROM CONVECTIVE ROLLING CELLS

4.1. Simplified structure of convective rolling cells

The velocity structure in a convective rolling cell can be simplified as a circular motion in, for example, (y, z) plane:

$$U_i = (0, V, W). \quad (41)$$

By use of a system of polar coordinates with the origin at the center of the cell, the velocities along y and z directions can be written as:

$$\begin{aligned} V &= -V_\theta \sin \theta, \\ W &= V_\theta \cos \theta, \end{aligned} \quad (42)$$

where the velocity in the θ direction can be expressed as:

$$V_\theta = r \Omega(r), \quad (43)$$

and the angular velocity Ω is regarded as only a function of the radius r to the center of the cell. It can be noticed that such a velocity structure satisfies the continuity equation of an incompressible fluid:

$$\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{\partial \Omega}{\partial \theta} = 0. \quad (44)$$

Based on such a simplified structure of convective rolling cells, the shear production rate P can be written as:

$$P = -\frac{\partial V}{\partial y} \overline{v^2} - \frac{\partial W}{\partial z} \overline{w^2} - \left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right) \overline{vw}. \quad (45)$$

If the Reynolds stress is approximated by the simple eddy viscosity model:

$$\overline{u_i u_j} = \frac{2}{3} k \delta_{ij} - \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad (46)$$

we obtain by substituting Eqs. (42) and (46) into Eq. (45) that:

$$P = \nu_t S^2, \quad (47)$$

where the shear of velocity S in the convective cell is defined by:

$$S = r \frac{\partial \Omega}{\partial r}. \quad (48)$$

It can be noticed that only the velocity shear contributes to the production of turbulence, while rigid body rotation has no effect on turbulence.

4.2. The temperature difference in a convective cell

The motion of a convective rolling cell will result in a temperature difference ΔT superimposed on the stationary temperature background. For a circular motion considered above, ΔT approximately satisfies the following equation in the steady state:

$$\frac{\lambda}{\rho c_p} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Delta T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Delta T}{\partial \theta^2} \right] - \frac{V_\theta}{r} \frac{\partial \Delta T}{\partial \theta} = \frac{N^2}{\beta g} V_\theta \cos \theta. \quad (49)$$

The first term on the left hand side of Eq. (49) describes heat dissipation due to radiation, and the second term gives the temperature difference of a displacement over a stratified fluid.

If the heat dissipation of radiation is ignored, the solution of Eq. (49) is:

$$\Delta T = -\frac{N^2}{\beta g} r \sin \theta = -\frac{N^2}{\beta g} z, \quad (50)$$

which is exactly what the standard mixing-length theory adopts. The general solution of Eq. (49) can thus be written in the complex form as:

$$\Delta T = i \frac{N^2}{\beta g} [r + B(r)] e^{i\theta}, \quad (51)$$

where B is assumed to be a function of the radius r . Substituting Eq. (51) into Eq. (49) and introducing a new independent variable:

$$\zeta = r \sqrt{i \frac{\rho c_p \Omega}{\lambda}}, \quad (52)$$

we obtain:

$$\zeta^2 \frac{\partial^2 B}{\partial \zeta^2} + \zeta \frac{\partial B}{\partial \zeta} - (1 + \zeta^2) B = 0. \quad (53)$$

This is a first-order modified Bessel equation, and its solution is:

$$B = B_0 I_1(\zeta) = B_0 \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \left(\frac{\zeta}{2} \right)^{2m+1}. \quad (54)$$

Therefore, the solution of temperature difference can be approximately written as:

$$\Delta T = i \frac{N^2 R_b}{\beta g} \left[\frac{r}{R_b} - B_0 I_1 \left(\frac{r}{R_b} \right) \right] e^{i\theta}. \quad (55)$$

This solution satisfies the boundary condition that $\Delta T = 0$ at $r = 0$. On the other side, ΔT should be zero at the surface of a convective rolling cell, which defines a typical size of convective rolling cells R_b as:

$$R_b = \sqrt{\frac{\lambda}{\rho c_p \Omega}}. \quad (56)$$

From the thermodynamic point of view, however, fluids undergoing the circular motion in a convective rolling cell operate as a heat engine in the convection zone, absorbing heat when it moves inward to higher temperature

region and releasing heat when it moves outward to lower temperature region and converting the difference of heat it absorbs and releases into its kinetic energy. As a result, the kinetic energy of its circular motion should be proportional to the heat involved in it:

$$V_\theta^2 \approx \Omega^2 R_b^2 \approx c_p \Delta T. \quad (57)$$

Considering Eqs. (55), (56), and (57) and assuming the turbulence in the fully local equilibrium described by Eq. (21), we approximate the averaged angular velocity of the circular motion of the convective cells by:

$$\Omega^{3/2} = \sqrt{\frac{\rho c_p}{\lambda}} \frac{\alpha H_p}{c'_{\mu} c'_{\epsilon 3} \tau^2}, \quad (58)$$

where α is a parameter similar as in the mixing-length theory.

4.3. The averaged shear of velocity in convective rolling cells

Usually, the unstable stratification in the stellar convection zone is fairly weak, and the circular motion reaches a steady state in a convective cell if the Reynolds stress balances the centrifugal force of circular motion. With the aid of Eqs. (7) and (42), it can be obtained:

$$\Omega^2 r \approx \left(\frac{\partial \overline{v^2}}{\partial y} + \frac{\partial \overline{vw}}{\partial z} \right) \cos \theta + \left(\frac{\partial \overline{w^2}}{\partial z} + \frac{\partial \overline{vw}}{\partial y} \right) \sin \theta. \quad (59)$$

Substituting the eddy viscosity model Eq. (46) into Eq. (59) and assuming that the shear of the circular motion is nearly a constant, we obtain:

$$\Omega^2 r \approx \frac{2}{3} \frac{\partial k}{\partial r} - 2 \left(\frac{\partial \nu_t}{\partial y} \frac{\partial V}{\partial y} \cos \theta + \frac{\partial \nu_t}{\partial z} \frac{\partial W}{\partial z} \sin \theta \right) + \left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right) \left(\frac{\partial \nu_t}{\partial z} \cos \theta + \frac{\partial \nu_t}{\partial y} \sin \theta \right). \quad (60)$$

It is interesting to note that an axially symmetric circular motion can induce an azimuth-dependent turbulence. Averaging Eq. (60) over all azimuthal angle θ and taking the averaged size of convective cells into account, we can evaluate the averaged kinetic energy of turbulence in a convective cell as:

$$k \approx \frac{3}{4} \Omega^2 r^2 \approx \Omega^2 R_b^2. \quad (61)$$

It can be seen that the kinetic energy is approximately partitioned more or less equally in the circular motion and in turbulence.

The averaged shear of velocity in the convective rolling cells can be regarded as the ratio of their typical velocity to their typical size. It should be recognized that the length scale of thermal convection is not only determined by the averaged size of convective cells, but also restricted by the macro-length scale of turbulence. When the size of the convective cell is larger than the macro-length of turbulence, the flow pattern of thermal convection is dominated by a cellular structure fully clogged up with numerous convective cells. If the macro-length of turbulence is larger than the averaged size of convective cells, the structure of an individual convective cell may be destroyed and several convective cells will merge into

a larger and somewhat chaotic flow pattern with a length scale comparable to the macro-length of turbulence.

Based on above arguments, we assume that the averaged shear of velocity in the convective cells can be approximated by:

$$S^2 \propto \frac{V_\theta^2}{R_b^2 + L^2}, \quad (62)$$

where L is the macro-length of turbulence. It can be noticed that:

$$\tau^2 S^2 \rightarrow \begin{cases} \tau^2 L^{-2} k & \text{if } R_b \ll L, \\ 0 & \text{if } R_b \gg L. \end{cases} \quad (63)$$

Accordingly, the velocity shear dominates the generation of turbulence for small convective cells, while the buoyancy production plays the major role to produce turbulence for large convective cells within them the temperature difference is able to become large enough. As a result, we suggest a model for the averaged shear of velocity in the convective cells as:

$$\tau^2 S^2 = \frac{1}{c_L + (L\Omega)^{-2} k} \quad (64)$$

with $c_L = (c_\mu)^{3/4}$.

4.4. Decay of the temperature difference along the vertical direction

As already discussed above, the convective rolling cells can effectively restrict the macro-length of turbulence, especially for the case when the macro-length of turbulence is much longer than the size of a convective cell. It is therefore interesting to understand how a plume-like vertical flow that is formed by merging several convective cells restricts the development of the macro-length of turbulence in the z direction, or, in other words, how far a convective rolling cell can move in the vertical direction.

The buoyancy is determined by the temperature difference of convective flow over the stationary structure. We assume that the velocity of a vertical plume is characterized by:

$$U_i = (0, 0, W), \quad (65)$$

where the velocity W in the z direction is only a function of distance r to axis z in the horizontal plane (x, y) as:

$$W = W(x, y) = W(r), \quad (66)$$

which is similar to the case that axis z is not placed at the center of a convective cell but at the boundary of two adjacent convective cells.

The equation of energy conservation Eq. (11) is now written as:

$$\frac{\lambda}{\rho c_p} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Delta T}{\partial r} \right) + \frac{\partial^2 \Delta T}{\partial z^2} \right] = \frac{\partial \Delta T}{\partial z} W + \frac{N^2}{\beta g} W. \quad (67)$$

The general solution of Eq. (67) is of the form:

$$\Delta T = -\frac{N^2}{\beta g} [z + Q(r) H(z)], \quad (68)$$

where the first term in the bracket of the right hand side is the well known mixing-length solution that has already been discussed for the convective cells in Eq. (50), and

the second term is what we are interested in at present. Substituting the formal solution Eq. (68) into the differential equation Eq. (67), we obtain:

$$\frac{\lambda}{\rho c_p} \left[\frac{H}{r} \frac{\partial}{\partial r} \left(r \frac{\partial Q}{\partial r} \right) + Q \frac{\partial^2 H}{\partial z^2} \right] = W Q \frac{\partial H}{\partial z}. \quad (69)$$

Equation (69) can be solved by the method of separation of variables:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dQ}{dr} \right) - \frac{Q}{R_b^2} = 0, \quad (70)$$

$$\frac{d^2 H}{dz^2} - \frac{1}{H_b} \frac{dH}{dz} + \frac{H}{R_b^2} = 0, \quad (71)$$

where

$$H_b = \frac{\lambda}{\rho c_p W}. \quad (72)$$

The solution of Eq. (70) is a zero order modified Bessel function, which describes an exponentially decay of velocity from the center of the plume and R_b measures therefore the width of the plume or the size of the convective cell.

With the aid of a new variable:

$$z = R_b \zeta, \quad (73)$$

Eq. (71) can be written as:

$$\frac{d^2 H}{d\zeta^2} - \chi \frac{dH}{d\zeta} + H = 0, \quad (74)$$

where

$$\chi = \frac{R_b}{H_b}. \quad (75)$$

This is a damped oscillation equation, and its solution in the weak damping limit ($\chi \ll 1$) is:

$$H = H_0 e^{-z/2H_b} \cos \left(\frac{z}{R_b} + \phi_0 \right). \quad (76)$$

It is interesting to note that the temperature difference along the z direction is a damped oscillating distribution, with the size of each unit being similar to a convective cell and a decaying distance of H_b .

On the other hand, the kinetic energy of a convective plume is simply a result of the work done by the buoyancy:

$$W^2 \approx -N^2 R_b^2, \quad (77)$$

because the buoyancy can only act within a distance comparable to the averaged size of convective cells. Considering Eqs. (56) and (72) and assuming that the turbulence is in the local equilibrium state, the decaying length of the temperature difference along a vertical plume can be approximately estimated as:

$$H_b^2 \approx c'_\mu c'_{\varepsilon 3} \Omega^2 \tau^2 R_b^2. \quad (78)$$

5. LOCAL SOLUTIONS OF THE k - ε MODEL

5.1. Choice of model parameter $c'_{\varepsilon 3}$

Usually, the model parameter $c_{\varepsilon 3}$ is taken to be a constant in the literature, and choices of its value disperse in a wide range from -1.4 to 1.45 (Baumert & Peters

2000). As already discussed in Sect. 3, the condition that $c'_{\varepsilon 3} > 1$ should be satisfied to ensure a state of local equilibrium existing in an unstably stratified region. Accordingly, we suggest a reasonable choice of $c'_{\varepsilon 3}$ for the stellar turbulent convection as:

$$c'_{\varepsilon 3} = 1 + \frac{c_\mu}{c_L - c_\mu + (\sqrt{c'_\mu c'_{\varepsilon 3}} \tau^2 \Omega^2)^{-1}}. \quad (79)$$

It is therefore easy to see that:

$$1 < c'_{\varepsilon 3} < 1 + \frac{c_\mu}{c_L - c_\mu}, \quad (80)$$

and with the aid of Eq. (19) that:

$$\frac{4}{3} < c_{\varepsilon 3} < \frac{4}{3} + \frac{1}{3} \frac{c_\mu}{c_L - c_\mu}. \quad (81)$$

Using the model of averaged velocity shear Eq. (64) and substituting Eq. (79) into Eq. (20), we obtain:

$$L^2 = \sqrt{c'_\mu c'_{\varepsilon 3}} \frac{k^3}{\varepsilon^2}, \quad (82)$$

which confirms the assumption that L in the model of averaged velocity shear is indeed the macro-length of turbulence in the state of fully local equilibrium. On the other hand, substituting Eq. (79) into the local equilibrium condition of buoyancy Eq. (21) and using Eq. (82), we obtain:

$$k \approx -\sqrt{c'_\mu c'_{\varepsilon 3}} L^2 N^2, \quad (83)$$

which is similar to the corresponding equation that has been used in the standard mixing-length theory.

5.2. Macro-length model in the mixing-length theory

In order to close the turbulent convection model discussed above, we have to specify the macro-length of convective turbulence. From Eq. (83) we know that the production of turbulent kinetic energy is a direct result of the mechanical work done by the buoyancy. The buoyancy comes from the temperature difference between the turbulently convective flow and its surroundings. As a result, the mechanical work done by the buoyancy is closely related to the amount of heat enclosed in the turbulent flow:

$$k \approx c_p \Delta T \approx -c_p \frac{N^2}{\beta g} L_b, \quad (84)$$

where L_b is an effective length scale over which the buoyancy efficiently does work on the turbulent flow in the stellar convection zone. By use of Eqs. (83) and (84) we obtain:

$$L^2 \approx H_p L_b. \quad (85)$$

If the effects of the convective rolling cells are not taken into account, it is reasonable to assume that L_b is just the macro-length of turbulence L itself. It can be seen from Eq. (85) that:

$$L \approx H_p, \quad (86)$$

which is exactly what the standard mixing-length theory assumes.

5.3. Macro-length model for stellar turbulent convection

One of the major effects of the convective rolling cells is to restrict the effective length for the buoyancy doing work on the turbulent flow. Assuming that L_b can be approximated by the decaying length of temperature difference along the z direction H_b , we obtain:

$$L^2 \approx \sqrt{c'_\mu c'_{\varepsilon 3}} \tau \Omega R_b H_p. \quad (87)$$

With the aid of Eqs. (56) and (87), we suggest a model for the macro-length of turbulence in the stellar convection zone as:

$$L^2 = \alpha H_p \sqrt{\tau \Omega \frac{\lambda \tau}{\rho c_p}}, \quad (88)$$

where α is a parameter similar as in the mixing-length theory. This macro-length model of turbulence will be applied to regions in both unstable and stable stratification.

5.4. Steady state solutions in the fully local equilibrium

In the local equilibrium state, it can be found from Eqs. (58), (82), and (88) that:

$$\tau^2 = \left(\frac{\rho c_p}{\lambda} \right)^2 \left(\frac{\alpha H_p}{c'_\mu c'_{\varepsilon 3}} \right)^4 \frac{(c'_\mu c'_{\varepsilon 3})^{3/2}}{y^3}. \quad (89)$$

Substituting Eqs. (58) and (89) into Eq. (79), the model parameter $c'_{\varepsilon 3}$ can be rewritten in the local equilibrium state as:

$$c'_{\varepsilon 3} = 1 + \frac{c_\mu}{c_L - c_\mu + y^{-1}}. \quad (90)$$

With the definition of a dimensionless quantity x as:

$$\frac{1}{x} = (c'_\mu c'_{\varepsilon 3})^{3/4} \frac{\rho c_p}{\lambda} \left(\frac{\alpha H_p}{c'_\mu c'_{\varepsilon 3}} \right)^2 \sqrt{\frac{\beta g T}{H_p} (\nabla_r - \nabla_{ad})}, \quad (91)$$

Eq. (40) can be rewritten as:

$$h = \frac{A_2 + \sqrt{A_2^2 + 4(1+y)c'_\mu y^{5/2}}}{2(1+y)} \quad (92)$$

$$\text{with } A_2 = c'_\mu c_T (xy)^{-2} + c'_\mu y^{5/2} - y - 1.$$

Combining Eqs. (21) and (37), the steady state solution in the fully local equilibrium is given by:

$$1 + y = (c_T + c'_{\varepsilon 3}) \frac{c'_\mu f}{x^2 y^2}. \quad (93)$$

By use of Eq. (32), it can be found that:

$$1 + y + \left(1 + \frac{c_T}{c'_{\varepsilon 3}} \right) c'_\mu y^{5/2} = (c_T + c'_{\varepsilon 3}) \frac{c'_\mu}{x^2 y^2}. \quad (94)$$

Furthermore, it can be found by combining Eqs. (93) and (94) that:

$$h = \left(1 + \frac{c_T}{c'_{\varepsilon 3}} \right) \frac{c'_\mu y^{5/2}}{1 + y}. \quad (95)$$

5.5. Comparisons with the standard mixing-length theory

According to the mixing-length theory, the macro-length of convection is characterized by the local pressure scale height, e.g. Eq. (86). As a result, we can approximate the turbulent time scale τ as:

$$\tau^2 \approx \left(\frac{\rho c_p}{\lambda}\right)^2 \left(\frac{\alpha H_p}{c'_\mu c'_{\varepsilon 3}}\right)^4 \frac{(c'_\mu c'_{\varepsilon 3})^{3/2}}{y^2}. \quad (96)$$

Substituting Eq. (96) into the local equilibrium condition Eq. (21), we obtain:

$$1 + y = (c_T + c'_{\varepsilon 3}) \frac{c'_\mu f}{x^2 y}. \quad (97)$$

The solution of Eq. (97) is:

$$2xy = \sqrt{x^2 + 4(c_T + c'_{\varepsilon 3})c'_\mu f} - x. \quad (98)$$

On the other hand, we assume that the time scale τ_h of the convective heat flux is proportional to the time scale of turbulence τ in Eq. (29). As a result, we obtain that:

$$(c_T + c'_{\varepsilon 3}) \frac{c'_\mu}{x^2 y} = 1 + y + \left(1 + \frac{c_T}{c'_{\varepsilon 3}}\right) c'_\mu y^2. \quad (99)$$

Substituting Eq. (97) into Eq. (99) we obtain:

$$x^3 y^3 = (c_T + c'_{\varepsilon 3}) \left(1 + \frac{c_T}{c'_{\varepsilon 3}}\right)^{-1} x(1 - f). \quad (100)$$

Substituting Eq. (100) into Eq. (98) we finally obtain:

$$\left(\sqrt{x^2 + 4(c_T + c'_{\varepsilon 3})c'_\mu f} - x\right)^3 = \frac{8(c_T + c'_{\varepsilon 3})}{1 + c_T/c'_{\varepsilon 3}} x(1 - f). \quad (101)$$

It can be seen that except for different model parameters, Eq. (101) is exactly the same as the result of the standard mixing-length theory (Cox & Giuli 1968):

$$\left(\sqrt{f + x^2} - x\right)^3 = \frac{8}{9} x(1 - f). \quad (102)$$

Our results are compared with that of the mixing-length theory in Fig. 1, in which we use h defined by Eq. (38) as the dependent variable in order to show more clearly the asymptotic behaviours of the two convection models. It can be seen that our model of Eq. (94) has the same asymptotic behaviour of a $-2/3$ decreasing index as that of the mixing-length theory when $x \rightarrow 0$, while declines more slowly with a $-5/2$ decreasing index than the mixing-length theory of a -4 decreasing index when $x \rightarrow \infty$.

6. A k - ω MODEL FOR STELLAR TURBULENT CONVECTION

Based on the model assumptions discussed in the preceding sections, the equation for the dissipation rate of the kinetic energy of turbulence can be written as:

$$\begin{aligned} & \frac{D\varepsilon}{Dt} - \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right) \\ &= \frac{3}{2} \left[P + \left(\frac{4}{3} + \frac{1}{3} (c'_{\varepsilon 3} - 1) \right) G \right] \frac{\varepsilon}{k} - 2 \frac{\varepsilon^2}{k} \end{aligned}$$

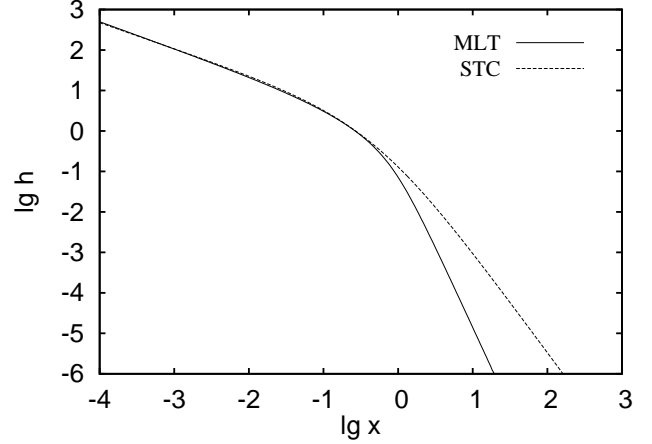


FIG. 1.— The efficiency of the convective heat transfer based on different convection models. The solid line (MLT) represents the result of the standard mixing-length theory, and the dashed line (STC) is the result of the steady state solution of the k - ε model in the fully local equilibrium.

$$\begin{aligned} &= \left(2 + \frac{1}{2} (c'_{\varepsilon 3} - 1)\right) \left[\frac{Dk}{Dt} - \frac{\partial}{\partial x_k} \left(c'_\mu \frac{k^2}{\varepsilon} \frac{\partial k}{\partial x_k} \right) \right] \frac{\varepsilon}{k} \\ &- \frac{P}{2\Omega^2 \sqrt{c'_\mu c'_{\varepsilon 3}}} \frac{\tau^{-2} - \sqrt{c'_\mu c'_{\varepsilon 3}} L^{-2} k}{c_L - c_\mu + (\sqrt{c'_\mu c'_{\varepsilon 3}} \tau^2 \Omega^2)^{-1}} \frac{\varepsilon}{k}. \end{aligned} \quad (103)$$

There are two facts to be noticed in Eq. (103). The first term on the right hand side of Eq. (103) contributes to an additional diffusion of turbulence as already pointed out by Pope (2000). The second term provides a local equilibrium state of Eq. (103) for not only the unstable stratification in the stellar convection zone but also the stable stratification in the overshooting regions. In consistency to the quasi-equilibrium assumption used to derive the macro-length model of turbulence, we introduce a model equation for the turbulence frequency ω as:

$$\begin{aligned} & \frac{D\omega}{Dt} - \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\omega} \frac{\partial \omega}{\partial x_i} \right) = \sqrt{c'_\mu c'_{\varepsilon 3}} \frac{k}{L^2} - \omega^2 \\ &= \frac{D^{1/3} H^{-4/3}}{\sqrt{c'_\mu c'_{\varepsilon 3}}} k \omega^{1/3} - \omega^2, \end{aligned} \quad (104)$$

where the turbulence frequency ω is defined by:

$$\omega = \frac{1}{\tau} = \frac{\varepsilon}{k}, \quad (105)$$

and two quantities of the stellar structure are defined by:

$$D = \frac{\rho c_p}{\lambda} \quad (106)$$

and

$$H = \frac{\alpha H_p}{c'_\mu c'_{\varepsilon 3}}. \quad (107)$$

We adopt Eq. (90) for the model parameter $c'_{\varepsilon 3}$ in accordance with the quasi-equilibrium assumption used to derive Eq. (104).

By use of Eqs. (37) and (40), the production rate of buoyancy can be written as:

$$G = \frac{2c'_\mu E y}{A_3 + \sqrt{A_3^2 + 4c'_\mu y^{5/2} c'_\mu c_T E \tau^2 y}} \frac{k^2}{\varepsilon} \quad (108)$$

with $A_3 = 1 + y + c'_\mu y^{5/2} - c'_\mu c_T E \tau^2 y$.

As a result, the kinetic energy equation of turbulence can be written as:

$$\begin{aligned} & \frac{Dk}{Dt} - \frac{\partial}{\partial x_i} \left(\nu_t \frac{\partial k}{\partial x_i} \right) \\ &= \frac{c'_\mu E D k^2 \omega}{\omega^3 + D k \omega^2 + c'_\mu (D k)^{5/2} \omega^{1/2} - c'_\mu c_T E D k} \frac{2}{1 + \sqrt{1 + A}} \\ & - \frac{(c_L - c_\mu) c'_\mu c'_\varepsilon H^{8/3} D^{1/3} \omega^{7/3} + k}{c_L c'_\mu c'_\varepsilon H^{8/3} D^{1/3} \omega^{7/3} + k} k \omega, \end{aligned} \quad (109)$$

where

$$A = \frac{4 c'_\mu (D k)^{5/2} \omega^{1/2} c'_\mu c_T E D k}{\left[\omega^3 + D k \omega^2 + c'_\mu (D k)^{5/2} \omega^{1/2} - c'_\mu c_T E D k \right]^2}.$$

7. APPLICATIONS OF THE k - ω MODEL IN DIFFERENT STELLAR MODELS

In order to compare the stellar turbulent convection model proposed in this paper with the standard mixing-length theory, we have applied both of them in calculations of stellar evolutionary models and compared the results. Our stellar evolutionary models are computed by a stellar evolution code `h04.f` originally described by Paczynski and Kozłowski and updated by Sienkiewicz (2004). The OPAL opacities (Rogers & Iglesias 1995; Iglesias & Rogers 1996) are used in the high temperature region and the opacities from molecules and grains (Alexander & Ferguson 1994) are used in the low temperature region. The Livermore Laboratory equation of state (Rogers et al. 1996) are adopted in our calculations. The nuclear reaction rates are updated according to Bahcall & Pinsonneault (1995) and Harris et al. (1983). We have modified the treatment of element diffusion according to Thoul et al. (1994).

7.1. Comparisons of solar models

Firstly, we apply the steady state solution (e.g. Eq. (94)) of the k - ω model in the fully local equilibrium to construct solar models. In order to calibrate our solar models to achieve the observed solar parameters (solar age $\tau_\odot = 4.57$ Gyr, solar mass $M_\odot = 1.9891 \times 10^{33}$ g, solar luminosity $L_\odot = 3.839 \times 10^{33}$ erg s $^{-1}$ (Bahcall & Pinsonneault 1995), and solar radius $R_\odot = 6.9566 \times 10^{10}$ cm (Haberreiter & Schmutz 2008)), we adjust initial helium abundance Y_0 and the mixing-length parameter α iteratively to ensure a relative accuracy of 10^{-4} for those solar parameters. The values of other parameters of the turbulent convection model are summarized in Table 2, as well as some basic properties of our solar models. In order for our solar models to have a depth of the convective envelope being in agreement with the result of helioseismic inversions ($R_{CZ} = 0.713 R_\odot$), the initial metal abundance has to be fixed to $Z_0 = 0.0208$ which results in a higher Z/X (0.0255) compared to the recent observations (0.0181) (Asplund et al. 2009).

The sound speed from the helioseismic inversion (Basu et al. 2009) is compared with those of our solar models with the mixing-length theory (MLT) and with the turbulent convection model of Eq. (94) (STC) in Fig. 2. It can be seen that the two solar models have almost identical sound speed profiles, except for in a region

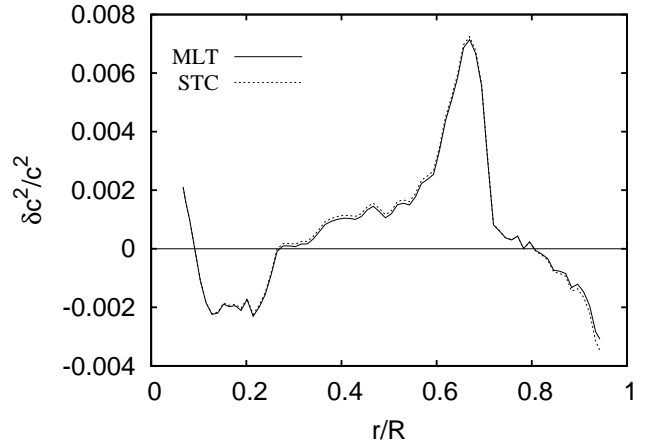


FIG. 2.— The sound speed difference between the helioseismic inversion and solar models based on different convection models. The solid line (MLT) represents the result of the standard mixing-length theory, and the dashed line (STC) is the result of the steady state solution of the k - ω model in the fully local equilibrium.

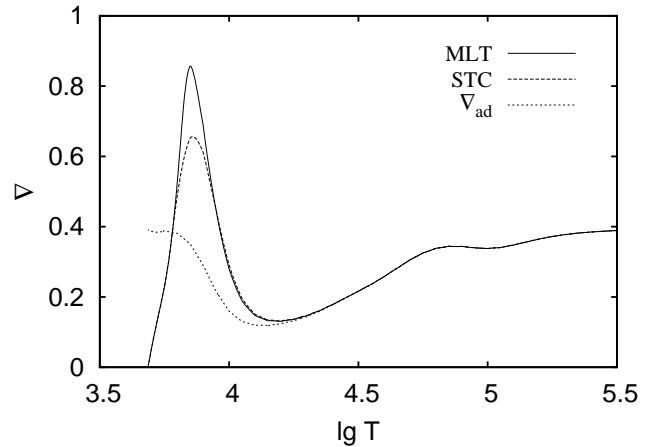


FIG. 3.— The temperature gradient in the upper envelope of solar models based on different convection models. The solid line (MLT) represents the result of the standard mixing-length theory, and the dashed line (STC) is the result of the steady state solution of the k - ω model in the fully local equilibrium. The adiabatic temperature gradient is given as the dotted line for reference.

about 5% below the solar photosphere. In order to show the difference of the two solar models just below the solar photosphere, the profiles of the temperature gradient ∇ of MLT and of STC are compared in Fig. 3, along with the adiabatic temperature gradient ∇_{ad} for reference. It can be seen that the turbulent convection model significantly reduces the temperature gradient just below the photosphere, which is a direct result of lower decreasing rate of the convective heat transfer efficiency of the turbulent convection model (Trampedach 2010).

The typical velocity of turbulent motions in the convection zone is compared in Fig. 4 for the two solar models with MLT and STC. It can be noticed that the two turbulence models predict similar maximum velocities of turbulence (about 3×10^5 cm s $^{-1}$) appeared just below the solar photosphere. But in most part of the convection zone, the STC predicts a typical velocity of turbulence (about 10^3 cm s $^{-1}$) one order of magnitude lower than what the MLT does (about 10^4 cm s $^{-1}$).

TABLE 2
PARAMETERS OF THE CONVECTION MODEL AND PROPERTIES OF SOLAR MODELS

Model	c_μ	c'_μ	c_T	α	Y_0	Y_S	Z_0	Z/X	$R_{CZ}(R_\odot)$
MLT				1.8750	0.27785	0.2474	0.0208	0.0255	0.7136
STC	0.1	0.008	0.5	1.5536	0.27785	0.2474	0.0208	0.0255	0.7138

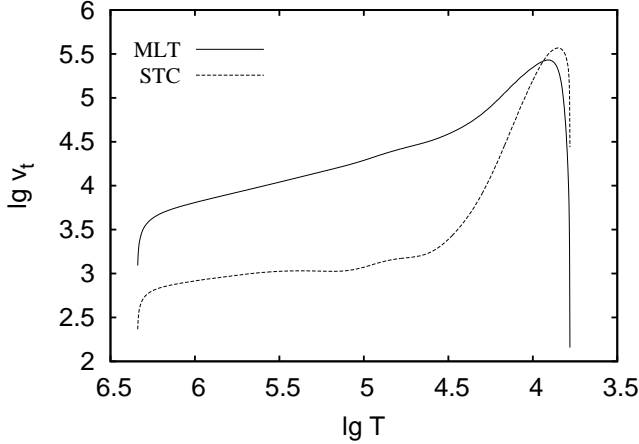


FIG. 4.— The distribution of the typical velocity of turbulence based on different convection models in the solar convection zone. The solid line (MLT) represents the result of the standard mixing-length theory, and the dashed line (STC) is the result of the steady state solution of the k - ω model in the fully local equilibrium.

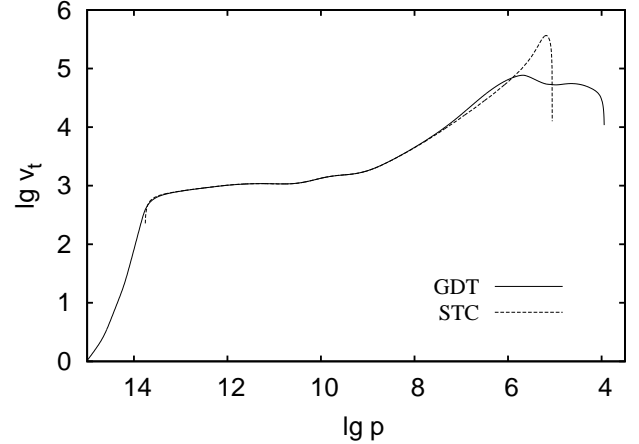


FIG. 5.— The comparison of the typical velocity of turbulence based on the steady state solution in the fully local equilibrium (STC) and on the full k - ω model including the transport effect with the gradient diffusion turbulence model (GDT).

Secondly, we apply the full k - ω model of Eqs. (104) and (109) to the whole solar interior including the convection zone and overshooting regions located below and above it, in order to study the transport effect of turbulence. It can be seen in Fig. 5 that the transport effect of the gradient diffusion turbulence model (GDT) is significant near the boundaries of the convection zone and in the overshooting regions. The typical velocity of turbulence roughly shows a linearly decaying law below the base of the solar convection zone, and according to results of different combinations of parameters values of the stellar turbulence model, the larger the parameter c'_μ is, the steeper the decaying law will be, which requires a smaller α to calibrate the model to the solar values. However, the turbulent velocity extends more or less constantly into the solar photosphere and lower atmosphere until they drop abruptly to the outer boundary conditions we have prescribed in the upper atmosphere. It is interesting to note that the typical velocity of turbulence is restricted below 1 km s^{-1} in the whole subphotospheric region due to the transport effect of turbulence, ensuring the thermally turbulent convection appearing there a subsonic motion as assumed in the stellar turbulence model. On the other hand, the typical length of turbulent motion remains almost a constant in the overshooting region below the base of the solar convection zone as shown in Fig. 6. Contrary to the local pressure scale height that decreases monotonically in the stellar envelope, the typical length of turbulence increases exponentially in the upper convection zone and overshooting region. This is a direct result of radiative dissipation that effectively increases the average size of convective rolling cells when the density drops rapidly in the solar convective envelope.

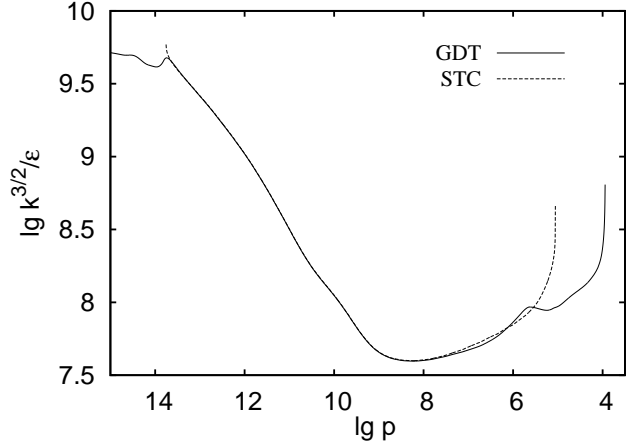


FIG. 6.— The comparison of the typical length of turbulence based on the steady state solution in the fully local equilibrium (STC) and on the full k - ω model including the transport effect with the gradient diffusion turbulence model (GDT).

7.2. Comparisons of stellar evolutionary models

We calculate a series of stellar evolution models of $0.6M_\odot$, $1.0M_\odot$, $1.5M_\odot$, $3.0M_\odot$, and $7.0M_\odot$, with the initial chemical abundance of $Y_0 = 0.27785$ and $Z_0 = 0.02$ and parameters of turbulence models given in Table 2. Element diffusion is not considered here for simplicity. The HR diagram is shown in Fig. 7 for the stellar models with different masses. It can be seen that the evolution tracks of stellar models based on the mixing-length theory (MLT) and on the model of stellar turbulent convection (STC) are almost identical, except that the STC results in a series of red giant branches with a little bit lower effective temperatures.

Profiles of the typical velocity of turbulence are com-

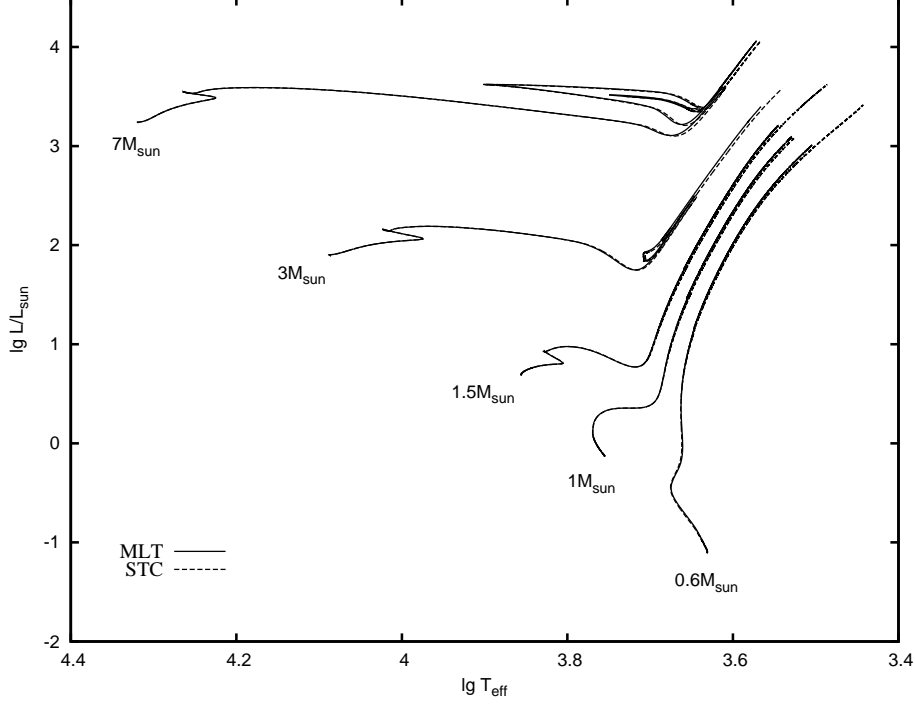


FIG. 7.— The HR diagram for stellar models of different masses based on different convection models. The solid line (MLT) represents the result of the standard mixing-length theory, and the dashed line (STC) is the result of the steady state solution of the $k-\omega$ model in the fully local equilibrium.

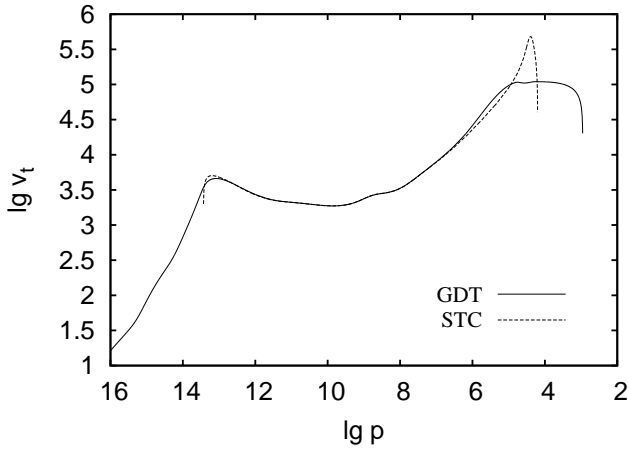


FIG. 8.— The comparison of the typical velocity of turbulence for an $1.0M_{\odot}$ RGB model. The solid line represents the result of the full $k-\omega$ model including the transport effect with the gradient diffusion turbulence model (GDT), and the dashed line is the result of the steady state solution in the fully local equilibrium (STC).

pared for $1.0M_{\odot}$ models based on STC and GDT at the RGB bump in Fig. 8, within which the envelope convection mostly penetrates inwardly into the stellar interior. It can be seen that the transport effect of turbulence is only significant below the base of the convective envelope, with an almost linearly decaying law similar to the solar case. On the other side, the turbulent velocity is effectively reduced in the upper convective envelope and diffused as nearly a constant value of 1 km s^{-1} into the stellar atmosphere. It is interesting to note in Fig. 9, however, that the typical length of turbulence varies only in one order of magnitude around 10^5 km in the whole

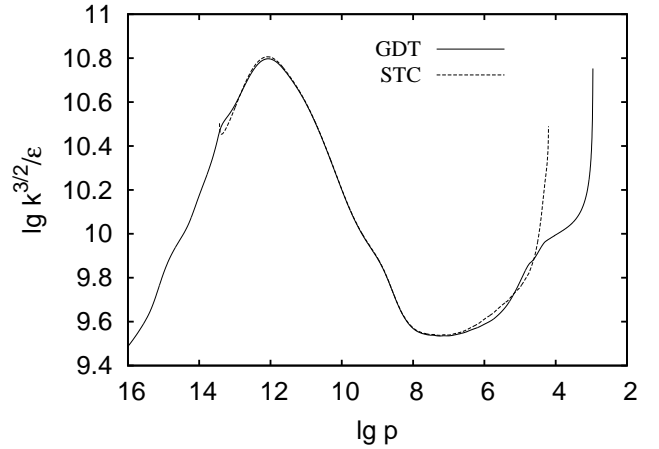


FIG. 9.— The comparison of the typical length of turbulence for an $1.0M_{\odot}$ RGB model. Others are the same as in Fig. 8.

convective envelope, decreasing linearly below the base of the convective envelope and increasing rapidly in the stellar atmosphere.

For a model of $7.0M_{\odot}$ in the main sequence, the transport effect of turbulence results in significant effect on the turbulent velocity in the convective core, the maximum velocity of GDT being only 80% as seen in Fig. 10 as that of STC. Outside the convective core, the turbulent velocity can be diffused into an overshooting region of about $0.5H_p$ wide. However, the turbulent velocity is decaying more and more rapidly in the overshooting region, resulting in a considerably shorter distance of fully mixing for elements outside the convective core. It can be seen in Fig. 11, however, that the typical length of turbulence remains almost as a constant in the convec-

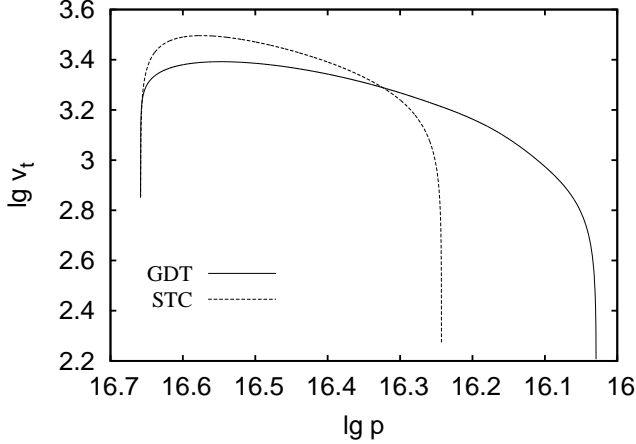


FIG. 10.— The comparison of the typical velocity of turbulence for a $7.0M_{\odot}$ main sequence model. The solid line represents the result of the full $k-\omega$ model including the transport effect with the gradient diffusion turbulence model (GDT), and the dashed line is the result of the steady state solution in the fully local equilibrium (STC).

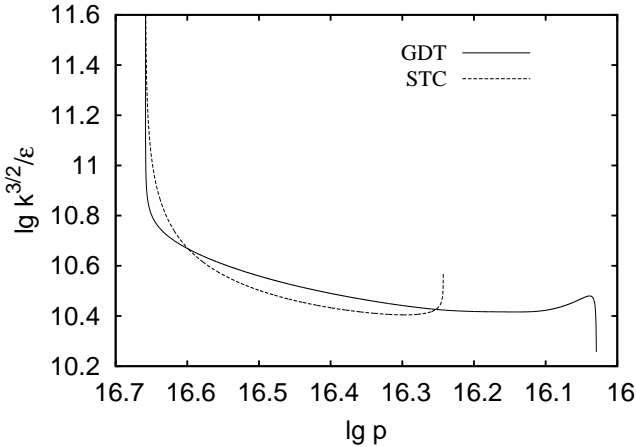


FIG. 11.— The comparison of the typical length of turbulence for a $7.0M_{\odot}$ main sequence model. Others are the same as in Fig. 10.

tive core and in the overshooting region, and its value is comparable to the size of the convective core. The typical length of turbulence sharply increases toward the stellar center due to the central boundary condition, which assumes the local equilibrium of turbulence at the center of a star.

Finally we apply the turbulence models of STC and GDT to a $3.0M_{\odot}$ model in the post-main-sequence evolutionary stage. According to the MLT, convection is underdeveloped in the envelope of such a star and turbulent motions only appear in two thin shells and have almost no effect on the structure of the stellar envelope. As seen in Fig. 12, however, the transport effect dominates the turbulent motion, and the turbulent velocity is diffused in the whole upper envelope upward to the stellar atmosphere and downward several local pressure scale heights below the lowest convective boundary. The resulted maximum velocity of turbulence is substantially reduced from STC's 16 km s^{-1} to GDT's 4 km s^{-1} . On the other hand, the typical length of turbulence remains almost a constant as seen in Fig. 13, and its value is

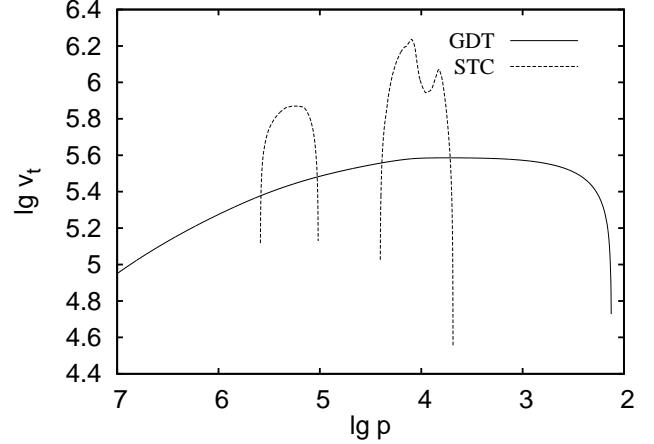


FIG. 12.— The comparison of the typical velocity of turbulence for a $3.0M_{\odot}$ sub-giant model. The solid line represents the result of the full $k-\omega$ model including the transport effect with the gradient diffusion turbulence model (GDT), and the dashed line is the result of the steady state solution in the fully local equilibrium (STC).

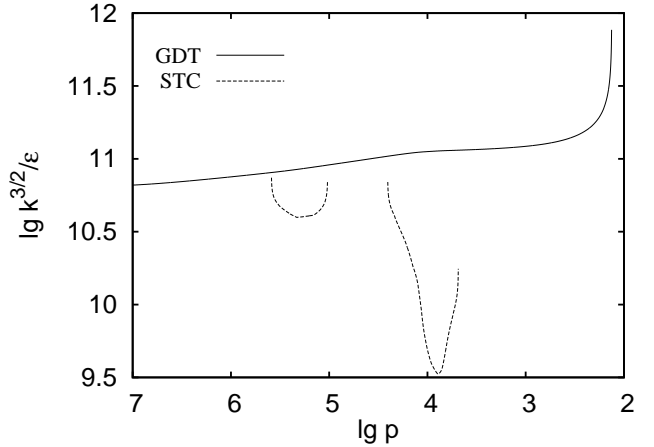


FIG. 13.— The comparison of the typical length of turbulence for a $3.0M_{\odot}$ sub-giant model. Others are the same as in Fig. 12.

significantly increased to be comparable to the width of the convective region being formed thereabout by the turbulent diffusion.

8. CONCLUSIONS AND DISCUSSIONS

Thermal convection is characterized in stars by convective rolling cells of different scales and a fully developed turbulence. Such semi-regular and large scale structures play a critical role in determining the macro properties of the convective motions, by both acting as thermal engines to transform heat into the kinetic energy to maintain their cellular configurations and producing the velocity shear within the convective rolling cells to generate turbulence. Due to great difficulties encountered both physically and numerically in dealing with such complex flows, turbulent convection models, which are based on fully hydrodynamic moment equations and include many physical properties of turbulence, have not yet taken the contributions from the convective rolling cells into account.

In order to investigate the qualitative properties of the convective rolling cells and to include their contributions

to the development of turbulence, we simplify their configurations by eddies rotating circle by circle with differential angular velocities, and approximately describe their motions by equations of mass and energy conservation. With the aid of advanced turbulence models, this simple approach enable us to estimate the average size of the convective rolling cells by solving for the temperature difference over the stationary temperature background, and to approximate the generation rate of turbulence due to the average shear of velocity in the convective rolling cells. On the other hand, the restriction from the heat conversion efficiency of such convective rolling cells working as individual thermal engines leads to a direct limit on the turbulent kinetic energy enclosed in them and an indirect evaluation of the typical length scale of turbulence in the fully local equilibrium. Based on these considerations and model assumptions thereof, we obtain the static solution of the standard k - ϵ model by properly introducing an important model parameter $c_{\epsilon 3}$, and then develop a k - ω model including the turbulent transport effect to describe stellar turbulent convection. Results of preliminary applications to the sun and other stars with different masses and in different evolutionary stages are in good agreement with those of the standard mixing-length theory and numerical simulations.

Being as an initial attempt to include the contributions from the convective rolling cells in the turbulent convection models, our approach have a lot of limitations and

incongruities. The most significant among them lies in the simplistic assumption that the convective rolling cells are of an eddy-like configuration, which is proved to be inadequate for the surface layer convection in the solar convective envelope where the scale height of the density stratification is much smaller than the average size of the convective rolling cells. The huge variation of density in the convective rolling cells results in great changes on the topology of the flows, i.e. a distinct up/down asymmetry characterized by hot and slow up-flows separated by cold and fast down-drafts (Stein & Nordlund 1989). On the other hand, the properties of the convective flow is determined by the physical conditions along the whole streamline that the fluids go through during a complete cycle of the convective motion. Such a nonlocal nature of the phenomenon, along with the huge range of variations of the density, significantly limits the validity of our approach by use of the local physical conditions to estimate the size of the convective rolling cells. Besides, the actual size and shape of the convective rolling cells may also be restricted by geometric conditions, for examples, in the stellar convective core or in thin convective shells of the stellar envelope with a thickness smaller than the local size of the convective rolling cells.

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